Local Distribution of rational points.

$X/\mathbb{Q}$ smooth, projective

$x \in X(\mathbb{R})$

① How close are rational pts. to x?
metric of Height $\leq B$
in fct. of B

② How dense are rational pts around x?
measure of height $\leq B$
in fct. of B.

1: Def. of approx. measures.

2: Def. of limit measures.

Classical Diophantine Approx.

Let $x \in \mathbb{R}$

$$a(x) = \sup \Big\{ a \in \mathbb{R} \;\Big|\; 0 < \left| x - \frac{p}{q} \right| \leq |q|^{-a} \quad (p,q) = 1 \text{ has inf. many solutions} \Big\}$$

The irrationality exp. of x

<u>Theorem</u> $a(x) = 1 \iff x \in \mathbb{Q}$.

$a(x) \geq 2 \iff x \notin \mathbb{Q}$

$a(x) = 2$ for a.e. $x \in \mathbb{R}$.

$a(x) \leq d$ if x is alg. of degree d. Dirichlet

$a(x) = 2$ if x is alg. but not $\in \mathbb{Q}$

(Roth's thm '57)

$a(e) = 2$

$a(\pi) \leq 14$ (Zudilin)

$x = \sum_{n=1}^{\infty} 10^{-n!}$ $\quad a(x) = \infty$ $\quad$ Liouville number

Approximation constants (Pagelot).

H 17a, b

X sm. proj. $/ \mathbb{Q}$

L ample line bundle on X

+ metrisation

$\iota : X \hookrightarrow \mathbb{P}^N$

Projective metric w.r.t. ι on $X(\mathbb{C})$

$$d(x,y) = \left(1 - \frac{\sum_{i=0}^{N} x_i \overline{y_i}}{\sum_{i=0}^{N} |x_i| \sum_{i=0}^{N} |y_i|} \right)^{1/2}$$

$[x_0 : \dots : x_N] = \iota(x)$

$[\quad = \iota(y)$

<u>Note</u> A different proj. embedding yield

an equivalent metric

$d_1 \leq \text{const}\, d_2 \qquad d_2 \leq \text{const}\, d_1.$

Choose $x \in X(\mathbb{R})$, $\emptyset \neq V \subseteq X$ Zariski open

Def: $c > 0$ and

$$\alpha(x, V) = \inf \left\{ \gamma > 0 \;\middle|\; \begin{array}{l} \text{There exists a sequence } (y_i)_{i=0}^{\infty} \\ \text{in } V(\mathbb{Q}) \setminus \{x\}, \text{ converging to } x \\ \text{such that } d(x, y_i)^{\gamma} \cdot H(y_i) \leq C \end{array} \right\}$$

approximation const. of x in V w.r.t. L

Ex: For $x \in \mathbb{P}^1(\mathbb{R})$ $\qquad \alpha(x, \mathbb{P}^1) = \dfrac{1}{a(x)}$

Replace $L \rightsquigarrow L^{\otimes d}$ $\qquad \alpha(x, V) \rightsquigarrow d \cdot \alpha(x, V)$

$x \in \mathbb{P}^1(\mathbb{R})$, $L = \mathcal{O}(d)$ then $\alpha(x, \mathbb{P}^1) = \dfrac{d}{a(x)}$

Obvious: $W \subseteq V \Rightarrow \alpha(x, W) \geq \alpha(x, V)$

Def (Pagelot)

$$\alpha_{ess}(x) = \sup_{V \subseteq X} \alpha(x, V)$$

<u>Example</u> $x \in \mathbb{P}^N(\mathbb{Q})$, $L = \mathcal{O}(d)$

$\alpha(x, \mathbb{P}^N) = \alpha_{ess}(x) = d.$

Same with $\mathbb{P}^{N_1} \times \cdots \mathbb{P}^{N_k}$.

<u>Example</u> X del Pezzo of deg 6

split over $\mathbb{Q}$. $\underset{p_1}{[1:0:0]}, \underset{p_2}{[0:1:0]}, \underset{p_3}{[0:0:1]}$

$x = [1:1:1]$, $L = \omega_X^{-1}$

$\alpha(x, X) = 2$

$\alpha(x, X \setminus Z) = 3 = \alpha_{ess}(x).$

$Z =$ str. transf. of the 3 lines L_1, L_2, L_3 joining x with $p_{1,2,3}$.

$\omega_X^{-1}|_{L_i} = \mathcal{O}(2)$

$\omega_X^{-1}|_{\substack{\text{any other} \\ \text{line} \neq L_1, L_2, L_3}} = \mathcal{O}(3)$

⚠ Not clear that $\alpha_{ess}(x) = \alpha(x, V)$

for some V

for some r.

Local Distribution.

$$\begin{pmatrix} \text{Nbhd of} \\ x \in X(\mathbb{R}) \end{pmatrix} \xrightarrow[\text{diffeo}]{\varsigma} T_x X \cong \mathbb{R}^n$$

$$x \longmapsto 0$$

Measure $\delta_{U,x,r,B}$

$$\int_{T_x X} f \, \delta_{U,x,r,B} = \sum_{\substack{y \in U(\mathbb{Q}) \\ H(y) \leq B}} f(B^{1/r} \varsigma(y))$$

$\forall f : T_x X \to \mathbb{R}$ comp. supported.

Expectation X Fano, toric, rat. connected. and appropriate coices of r, γ, B $x \in X(\mathbb{R} \cap \bar{\mathbb{Q}})$, the limit measure

$$\lim_{B\to\infty} \frac{\delta_{U,x,r,B}}{B^{\gamma} \log(B)^{\beta}}$$ exists.

Interesting : $r = \alpha_{ess}(x)$ or other values $\alpha(x, V)$.

Example (Pagelot) $x = [0:1] \in \mathbb{P}^1$

$\alpha_{ess}(x) = 1$

$r < 1$
$$\lim_{B\to\infty} \int f \delta_{\mathbb{P}^1, x, r, B} = f(0)$$

$r > 1$
$$\lim_{B\to\infty} B^{-2+\frac{1}{r}} \int f \delta_{\mathbb{P}^1, x, r, B} = \frac{3}{\pi^2} \int_{-\infty}^{\infty} f(t) d$$

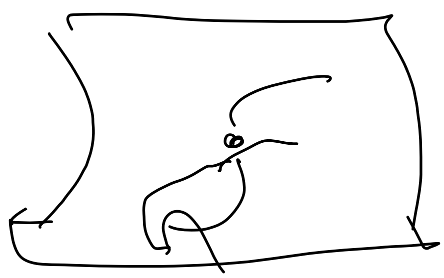
$r = 1$.
$$\lim_{B\to\infty} B^{-1} \int f \delta_{\mathbb{P}^1, x, r, B} = \int_{-\infty}^{\infty} f(t) \frac{\sigma(t)}{t^2} dt$$

$$\sigma(t) = \sum_{n \leq |t|} \# (\mathbb{Z}/n\mathbb{Z})^{r}$$

<u>Other Results</u> Know limit measures for split del Pezzo surf. of degree 6, 7, 8, 9.

$\alpha(x, U)$

Suppose $x \in X(\mathbb{Q})$ and suppose that $\exists$ rational curve in X through x.



<u>Then</u> there exists (another) rat. curve $C \subseteq X$ with $x \in C(\mathbb{Q})$

and $\qquad \alpha(x, X) \doteq \alpha(x, C).$

X = simple ab. variety of dim ≥ 2

$x \in X(\mathbb{Q})$ not torsion. $\quad x$ lim pt of $\mathbb{Z} \cdot x \subseteq X(\mathbb{Q})$

Any curve on X genus ≥ 2